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(Bosonic)mass meets (extrinsic)curvature

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Abstract

In this paper, we discuss the mechanism of spontaneous symmetry breaking from the point view of vacuum pairs, considered as ground states of a Yang-Mills–Higgs gauge theory. We treat a vacuum as a section in an appropriate bundle that is naturally associated with a minimum of a (general) Higgs potential. Such a vacuum spontaneously breaks the underlying gauge symmetry if the invariance group of the vacuum is a proper subgroup of the gauge group. We show that each choice of a vacuum admits to geometrically interpret the bosonic mass matrices as "normal" sections. The spectrum of these sections turns out to be constant over the manifold and independent of the chosen vacuum. Since the mass matrices commute with the invariance group of the chosen vacuum one may decompose the Hermitian vector bundles which correspond to the bosons in the eigenbundles of the bosonic mass matrices. This decomposition is the geometrical analog of the physical notion of a "particle multiplet". In this sense, the basic notion of a "free particle" also makes sense within the geometrical context of a gauge theory, provided the gauge symmetry is spontaneously broken by some vacuum.

We also discuss the Higgs–Kibble mechanism ("Higgs Dinner") from a geometrical point of view. It turns out that the "unitary gauge", usually encountered in the context of discussing the Higgs Dinner, is of purely geometrical origin. In particular, we discuss rotationally symmetric Higgs potentials and give a necessary and sufficient condition for the unitary gauge to exist. As a specific example, we discuss in some detail the electroweak sector of the Standard Model of particle physics in this context.

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1. Introduction

We consider the mechanism of spontaneous symmetry breaking from a geometrical viewpoint. For this we treat "elementary particles" as (a gauge equivalence class of) Hermitian vector bundles over an orientable spacetime $(\mathcal{M}, g_{\mathrm{M}})$. Here, g_{M} is an arbitrary but fixed (pseudo)metric (see also, for instance [3]). The possible states of the particles are geometrically represented as sections of the corresponding vector bundles. The gauge group is then given by the subgroup of automorphisms of these bundles which induce the identity map on the spacetime manifold. In the present paper, we shall focus on bosons only. We also restrict ourselves to the case of a pure Yang-Mills–Higgs gauge theory. We characterize such a gauge theory by a specific set of geometrical data. In particular, the gauge group will be identified with the gauge group of a principal G-bundle $\mathcal{P}(\mathcal{M}, G)$. From the given data we build two bundles, one of which geometrically represents the Higgs boson. Correspondingly, we call this bundle the "Higgs bundle". The other bundle, which we call the "orbit bundle", is a certain sub-bundle of the Higgs bundle. Sections $\mathcal{V}$ of the orbit bundle physically represent possible ground states of the Higgs boson. In fact, these sections minimize the Higgs potential which we also treat as a globally defined object. Accordingly, we call such a section $\mathcal{V}$ a "vacuum section".

From a geometrical perspective, a vacuum section is in one-to-one correspondence with an H-reduction of $\mathcal{P}(\mathcal{M}, G)$. Here, the (closed) subgroup $H \subset G$ corresponds to the stabilizer group of some minimum $\mathbf{z}_0$ of a general Higgs potential V_{H}. Therefore, this subgroup gives rise to the invariance group of the "vacuum" which is defined by the section $\mathcal{V}$ (i.e. by a ground state of the Higgs boson). As usual, if the invariance group is a subgroup of the gauge group, we call the latter spontaneously broken by the vacuum.

We then introduce the notion of "vacuum pairs". They consist of vacuum sections $\mathcal{V}$ and connections Ξ on the Higgs bundle ξ_{H} which are compatible with $\mathcal{V}$. Let ∂ be the covariant derivative with respect to Ξ. Then, the vacuum pair $(\partial, \mathcal{V})$ geometrically generalizes $(\mathrm{d}, \mathbf{z}_0)$ usually considered in particle physics. Of course, the latter makes sense only if $\mathcal{P}(\mathcal{M}, G)$ is supposed to be the trivial principal G-bundle $\mathcal{M} \times G \xrightarrow{\mathrm{pr}_1} \mathcal{M}$. In general, there exist gauge inequivalent vacuum pairs (also in the case where $\mathcal{P}(\mathcal{M}, G)$ is supposed to be trivial). We will show that, if spacetime is simply connected, then all vacuum pairs are gauge equivalent to the canonical one.

Since the ground states of the Higgs boson are treated as a globally defined objects (sections) the physical decomposition of the Higgs boson into the Goldstone and the physical Higgs boson is geometrically reflected by a $\mathbb{Z}_2$-grading of the reduced Higgs bundle. Likewise, with respect to a vacuum pair, the reduced adjoint bundle, which geometrically represents a gauge boson, splits into two real vector bundles. These represent the residual gauge boson and a massive vector boson. In fact, the rank of the vector bundle representing the massive gauge boson equals the rank of the "Goldstone bundle". This gives rise to a geometrical description of the known Higgs–Kibble mechanism (i.e. to the so-called "Higgs Dinner").

The description of the mechanism of spontaneous symmetry breaking in terms of an H-reduction of a given principal G-bundle is well known and can be found, for instance, in [1,2] or [9]. We also refer to [10] and the corresponding references therein like, e.g. [7]. For a good exposition of the fiber bundle description of gauge theories that is between "mathematics and physics", we refer to [11].

Though clear from a mathematical point of view, the geometrical description of the ground states of the Higgs boson in terms of vacuum sections seem to be less known. The notion of vacuum section is physically intuitive and permits to treat the bosonic mass matrices as sections as well. We show that the mass matrices can be regarded as "normal vector fields" of specific sub-manifolds and thus are related to the extrinsic curvature of these sub-manifolds. The bundles representing the physical Higgs boson and the massive gauge boson can be decomposed into the eigenbundles of the (non-trivial part of the) respective bosonic mass matrices. This expresses the notion of "particle multiplets" in purely geometrical terms without reference to any gauge. In particular, the proposed setup allows to geometrically describe "free particles" within gauge theories. The notion of vacuum pairs also gives rise to a geometrical understanding of the unitary gauge. For a specific class of Higgs potentials, we present a necessary and sufficient condition for this gauge to exist. This class of potentials includes the Higgs potential postulated in particle physics. As a specific example, we discuss the unitary gauge in the case of the electroweak sector of the Standard Model from the geometrical point of view presented here.

The aim of the paper is to emphasize the geometry which underlies spontaneously broken gauge theories. In particular, we want to stress how the notion of mass might be related to the topology of spacetime if the mechanism of spontaneous symmetry breaking is treated from a global point of view. The motivation for the present work might be best summarized by quoting a famous statement by H. Weyl:

"*Every physical quantity will be represented by a geometrical object*".

One may ask for the geometrical objects representing "free particles" and their corresponding "masses" within the geometrical frame of (spontaneously broken) gauge theories. To geometrically consider "particles" as (gauge equivalent) vector bundles and states as sections mainly results from the well-known circumstance that a general gauge group seems to have no physical realization. In particular, a (local) trivialization of a general principal G-bundle $\mathcal{P}(\mathcal{M}, G)$ has no physical counterpart.[1] Likewise, a specific gauge condition cannot be physically realized, in general. Therefore, any reference to some gauge (or local trivialization) should be avoided in a geometrical description of "particles" and their properties like "mass" and "charge". For this reason, it seems inadequate to geometrically identify particles with sections and "free particles" with "components" of the typical fiber with respect to some (local) trivialization. Since $\mathcal{P}(\mathcal{M}, G)$ has no direct physical meaning, its definite topological structure can only be determined by additional physical arguments. For instance, if there were no (massless) gauge boson in nature, then $\mathcal{P}(\mathcal{M}, G)$ would have to be trivial. Or, as we will show, if spacetime is supposed to be simply connected, then vacuum pairs exist if and only if $\mathcal{P}(\mathcal{M}, G)$ is trivial. To put emphasis on a possible relation between the topology of $\mathcal{M}$ and $\mathcal{P}(\mathcal{M}, G)$ on the basis of spontaneously broken gauge theories is a matter of concern of this paper.

The paper is organized as follows. In Section 2, we introduce the notion of vacuum pairs and discuss the bosonic mass matrices as sections. In Section 3, we consider the

[1] This is quite different from the case of the theory of general relativity. Not only does the frame bundle of spacetime have a physical meaning but in relativity there also exist physical quantities like, e.g. energy and momentum that can be defined only with respect to some reference frame (local trivialization of the frame bundle). This should not be confounded with the assumption that any physical statement should be frame-independent.

Higgs–Kibble mechanism from a geometrical perspective and discuss the unitary gauge, as well as the notion of "free particles" within the context of gauge theories. In Section 4, we geometrically interpret the bosonic mass matrices as "normal sections" of specific sub-manifolds defined by a vacuum. Finally, in Section 5, we discuss the existence of the unitary gauge in the case of the structure group of the electroweak sector of the Standard Model from the geometrical viewpoint presented in this paper. We finish with a brief summary and outlook.

2. Vacuum pairs and the bosonic mass matrices

The aim of this section is to geometrically formulate the physical notion of a "vacuum" within the framework of gauge theories. In doing so, the basic notion we have to introduce is that of an "orbit bundle". To start with, we denote by $(\mathcal{M}, g_{\mathrm{M}})$ a smooth orientable (pseudo)Riemannian manifold. Topologically, $\mathcal{M}$ is supposed to be a Hausdorff space that is paracompact and (pathwise) connected. Since in this paper a (pseudo)metric g_{M} is assumed to be fixed we simply refer to $\mathcal{M}$ as "spacetime".

A *Yang-Mills–Higgs gauge theory* is specified by the data $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{H}}, V_{\mathrm{H}})$. Here, $\mathcal{P}(\mathcal{M}, G)$ denotes a principal G-bundle $P \stackrel{\pi_P}{\to} \mathcal{M}$, where the structure group G is assumed to be a compact, real and semi-simple Lie group with Lie algebra Lie(G). The corresponding Killing form is denoted by κ_{G}. The unitary (orthogonal) representation $\rho_{\mathrm{H}} : G \to \mathrm{Aut}(\mathbb{K}^N)$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ is assumed to be faithful. The smooth real valued function $V_{\mathrm{H}} \in \mathcal{C}^\infty(\mathbb{K}^N, \mathbb{R})$ is supposed to be bounded from below and to be G-invariant. Moreover, transversally to each orbit of minima of V_{H} the Hessian of this function is positive definite. In this case V_{H} is called a *general Higgs potential*.

An immediate consequence of the above given data is the existence of a specific Hermitian vector bundle ξ_{H}:

$$\pi_{\mathrm{H}} \colon E_{\mathrm{H}} \colon P \times_{\rho_{\mathrm{H}}} \mathbb{K}^N \to \mathcal{M}. \tag{1}$$

We call this bundle the *Higgs bundle*. It is considered to be the geometrical analog of the Higgs boson. Accordingly, states of the Higgs boson are identified with sections $\Phi \in \Gamma(\xi_{\mathrm{H}})$.

Because of its G-invariance a general Higgs potential induces a smooth mapping (also denoted by V_{H}):

$$V_{\mathrm{H}} \colon \Gamma(\xi_{\mathrm{H}}) \to \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}), \qquad \Phi \mapsto \phi^* V_{\mathrm{H}}. \tag{2}$$

Here, $\phi \in \mathcal{C}^\infty_{\rho\text{-eq}}(P, \mathbb{K}^N) \simeq \Gamma(\xi_{\mathrm{H}})$ is the ρ_{H}-equivariant mapping, which corresponds to the state Φ of the Higgs boson, i.e. $\Phi(x) = [(p, \phi(p))]|_{p \in \pi_P^{-1}(x)}$. Then, $\phi^* V_{\mathrm{H}}$ is defined by $\phi^* V_{\mathrm{H}}(x) := V_{\mathrm{H}}(\phi(p))|_{p \in \pi_P^{-1}(x)}$. The corresponding action functional is denoted by[2]:

$$\mathcal{V}_{\mathrm{H}} \colon \Gamma(\xi_{\mathrm{H}}) \to \mathbb{R}, \qquad \Phi \mapsto \langle \phi^* V_{\mathrm{H}}, 1 \rangle. \tag{3}$$

[2] Of course, this functional is only well defined if the states satisfy suitable boundary conditions (or $\mathcal{M}$ is supposed to be compact).

Here, $\langle\cdot,\cdot\rangle$ denotes the usual pairing on $\Omega(\mathcal{M},E) := \Gamma(\xi_{\Lambda(T^*\mathcal{M})}\otimes\xi)$, where $\xi_{\Lambda(T^*\mathcal{M})}$ is the Grassmann bundle and ξ any Hermitian vector bundle over $\mathcal{M}$ with total space E. We call the action (3) the *global Higgs potential*.

Let $\mathcal{A}(\xi_{\rm H})$ be the affine set of all associated connections on the Higgs bundle. The *Yang-Mills–Higgs action*, based on the data $(\mathcal{P}(\mathcal{M},G),\rho_{\rm H},V_{\rm H})$, then reads

$$\begin{aligned}&\mathcal{I}_{\rm YMH}\colon \mathcal{A}(\xi_{\rm H})\times\Gamma(\xi_{\rm H})\to\mathbb{R},\\ &(A,\Phi)\mapsto s\langle F_{\rm A},F_{\rm A}\rangle+\langle\partial_{\rm A}\Phi,\partial_{\rm A}\Phi\rangle+s\mathcal{V}_{\rm H}(\Phi)\equiv\mathcal{I}_{\rm YM}(A)+\mathcal{I}_{\rm H}(A,\Phi).\end{aligned}\tag{4}$$

Here, $s=\pm1$ depends on the signature of $g_{\rm M}$ and $F_{\rm A}\in\Gamma(\xi_{\Lambda^2(T^*\mathcal{M})}\otimes\xi_{{\rm ad}(P)})$ is the field strength with respect to the connection A, and $\partial_{\rm A}$ is the corresponding covariant derivative on $\Gamma(\xi_{\rm H})$. By $\xi_{{\rm ad}(P)}$ we mean the *adjoint bundle*:

$$\pi_{\rm ad}\colon {\rm ad}(P) := P\times_G {\rm Lie}(G)\to\mathcal{M}.\tag{5}$$

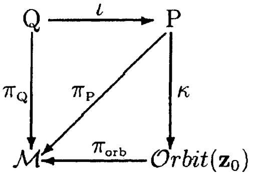
The *gauge group* of $\mathcal{P}(\mathcal{M},G)$ is denoted by $\mathcal{G}$. As usual we identify $\mathcal{G}$ with $\mathcal{C}^\infty_{\text{Ad-eq}}(P,G)\simeq{\rm Aut}_{\rm eq}(P)$. Here, the latter denotes the subgroup of right equivariant automorphisms on P which induce the identity on $\mathcal{M}$.

Besides the Higgs bundle and the Yang-Mills–Higgs action, there is still another geometrical object that is naturally associated with the data specifying a Yang-Mills–Higgs gauge theory. For this, let $\mathbf{z}_0\in\mathbb{K}^N$ be a minimum of $V_{\rm H}$. We denote by, respectively, ${\rm orbit}(\mathbf{z}_0)\subset\mathbb{K}^N$ and $I(\mathbf{z}_0)\subset G$ the orbit associated with $\mathbf{z}_0$ and the isotropy group of the minimum. Up to conjugation, there is a unique closed subgroup $H\subset G$ such that $H\simeq I(\mathbf{z}_0)$ and ${\rm orbit}(\mathbf{z}_0)\simeq G/H$. Thus, up to equivalence (within the category of bundles) a minimum $\mathbf{z}_0$ is associated with a specific sub-bundle $\xi_{{\rm orbit}(\mathbf{z}_0)}\subset\xi_{\rm H}$ of the Higgs bundle:

$$\pi_{\rm orb}\colon \mathcal{O}{\rm rbit}(\mathbf{z}_0) := P\times_{\rho_{\rm orb}}{\rm orbit}(\mathbf{z}_0)\to\mathcal{M}.\tag{6}$$

Here, $\rho_{\rm orb} := \rho_{\rm H}|_{{\rm orbit}(\mathbf{z}_0)}$. We call this fiber bundle the *orbit bundle* with respect to the minimum $\mathbf{z}_0$. Notice that sections $\mathcal{V}\in\Gamma(\xi_{{\rm orbit}(\mathbf{z}_0)})$ of the orbit bundle can also be considered as sections of the Higgs bundle and thus as specific states of the Higgs boson. Since these states minimize the global Higgs potential (3) we call them *vacuum sections*.

As a closed subgroup of the structure group G, the group H also acts on P from the right and therefore makes $P\stackrel{\kappa}{\to}\mathcal{O}{\rm rbit}(\mathbf{z}_0)$ a principal H-bundle. As a consequence, every vacuum section corresponds to an $I(\mathbf{z}_0)$-reduction of $\mathcal{P}(\mathcal{M},G)$. This means that $\mathcal{V}\in\Gamma(\xi_{{\rm orbit}(\mathbf{z}_0)})$ determines (up to equivalence) a unique principal H-bundle $\mathcal{Q}(\mathcal{M},H)$ together with an embedding $Q\stackrel{\iota}{\to}P$, such that the following diagram commutes:



We call $(\mathcal{Q},\iota)$ a *vacuum* with respect to a minimum $\mathbf{z}_0$. Notice that a vacuum also determines a vacuum section by putting $\mathcal{V}(x) := [(\iota(q),\mathbf{z}_0)]|_{q\in\pi_Q^{-1}(x)}$ for all $x\in\mathcal{M}$. Therefore, there is a one-to-one correspondence between the ground states of the Higgs boson and the vacua (for instance, cf. Chapter 1.5, Proposition 5.6 in [6]). We call the reduced gauge group $\mathcal{H} \simeq \mathrm{Aut}_{\mathrm{eq}}(Q)$ the *invariance group of the vacuum* $(\mathcal{Q}, \iota)$. A Yang-Mills–Higgs gauge theory is called *spontaneously broken* with respect to a vacuum $(\mathcal{Q}, \iota)$ if the invariance group of the latter is a proper subgroup of the original gauge group $\mathcal{G}$. The gauge symmetry is called maximally broken by the vacuum if its invariance group is trivial. Note that in this case $\mathcal{P}(\mathcal{M}, G)$ must necessarily be trivial. However, the H-reduction of a trivial principal G-bundle can be nontrivial. In general, we call a vacuum $(\mathcal{Q}, \iota)$ trivial iff $\mathcal{Q}(\mathcal{M}, H)$ is equivalent to the trivial principal H-bundle $\mathcal{M} \times H \stackrel{\mathrm{pr}_1}{\to} \mathcal{M}$. Notice that there is a distinction between a trivial vacuum and the case where the gauge symmetry is completely broken, i.e. $H = \{e\}$.

Though $\mathcal{Q}(\mathcal{M}, H)$ is not equivalent to the original principal G-bundle, every G-associated fiber bundle is equivalent to its H-reduction. More precisely: let $\xi_E : E := P \times_\rho W \stackrel{\pi_E}{\to} \mathcal{M}$ be a G-associated fiber bundle with typical fiber W and representation $G \stackrel{\rho}{\to} \mathrm{Diff}(W)$. Moreover, let $\xi_{E,\mathrm{red}}$ be the corresponding reduced fiber bundle with respect to a vacuum $(\mathcal{Q}, \iota)$, i.e. $\pi_{E,\mathrm{red}} : E_{\mathrm{red}} := Q \times_{\rho_{\mathrm{red}}} W \to \mathcal{M}$. Here, $\rho_{\mathrm{red}} := \rho|_{\mathrm{H}}$. Then, we have $\xi_E \simeq \xi_{E,\mathrm{red}}$. The corresponding bundle isomorphism (over the identity on $\mathcal{M}$) is given by the diffeomorphism:

$$E_{\mathrm{red}} \to E, \qquad [(q, \mathbf{w})] \mapsto [(\iota(q), \mathbf{w})]. \tag{7}$$

This will be crucial in what follows. For instance, every vacuum section corresponds to a constant section (also denoted by $\mathcal{V}$) in the reduced Higgs bundle $\xi_{\mathrm{H,red}}$ defined by the appropriate vacuum $(\mathcal{Q}, \iota)$:

$$\mathcal{V}: \mathcal{M} \to E_{\mathrm{H,red}}, \qquad x \mapsto [(q, \mathbf{z}_0)]|_{q \in \pi_Q^{-1}(x)}. \tag{8}$$

This geometrically generalizes the following situation usually encountered in physics. Let $\mathcal{P}(\mathcal{M}, G)$ be the trivial principal G-bundle $\mathcal{M} \times G \stackrel{\mathrm{pr}_1}{\to} \mathcal{M}$. In this case, the orbit bundle with respect to a minimum $\mathbf{z}_0$ has a canonical section given by the constant mapping (also denoted by $\mathbf{z}_0$):

$$\mathbf{z}_0 : \mathcal{M} \to \mathcal{M} \times \mathrm{orbit}(\mathbf{z}_0), \qquad x \mapsto (x, \mathbf{z}_0). \tag{9}$$

In this case, the corresponding vacuum is simply given by the inclusion:

$$\iota: \mathcal{M} \times H \hookrightarrow \mathcal{M} \times G, \qquad (x, h) \mapsto (x, h). \tag{10}$$

Clearly, (8) generalizes (9) to geometrical situations where no specific assumption on $\mathcal{P}(\mathcal{M}, G)$ has been made. As we have already mentioned, even in the case where $\mathcal{P}(\mathcal{M}, G)$ is trivial there might exist nontrivial vacua that cannot be gauge equivalent to the canonical vacuum (10). Therefore, it seems appropriate to deal with the more general situation described by (8).

A vacuum section (8) defines a constant section of the reduced Higgs bundle. It is also covariantly constant with respect to any connection $A \in \mathcal{A}(\xi_{\mathrm{H,red}})$. The latter denotes the affine set of associated connections on the reduced Higgs bundle. Thus, with respect to a vacuum $(\mathcal{Q}, \iota)$ there exists a distinguished affine subset of connections on $\mathcal{P}(\mathcal{M}, G)$.[3]

Definition 2.1. A connection A on $\mathcal{P}(\mathcal{M}, G)$ is called to be "compatible" with a vacuum section $\mathcal{V}$ if it also defines a connection on $\mathcal{Q}(\mathcal{M}, H)$.

Notice that a connection A on $\mathcal{P}(\mathcal{M}, G)$ is compatible with $\mathcal{V}$, iff its connection form $\omega \in \Omega^1(P, \mathrm{Lie}(G))$ satisfies $\iota^*\omega \in \Omega^1(Q, \mathrm{Lie}(H))$.

Definition 2.2. A Yang-Mills–Higgs pair $(A, \Phi) \in \mathcal{A}(\xi_{\mathrm{H}}) \times \Gamma(\xi_{\mathrm{H}})$ is called a "vacuum pair" provided $\Phi \equiv \mathcal{V}$ is a vacuum section and $A \equiv \Xi$ is induced by a flat connection on $\mathcal{P}(\mathcal{M}, G)$, which is compatible with $\mathcal{V}$. The corresponding covariant derivative on $\Gamma(\xi_{\mathrm{H}})$ is denoted by ∂.

A vacuum $(\mathcal{Q}, \iota)$ defines a minimum of the energy of the Higgs boson. In fact, let us denote by $\wp^{\mathrm{H}}$ the horizontal projector of a reducible connection A on $\mathcal{P}(\mathcal{M}, G)$. It induces a corresponding horizontal projector (and thus a connection) on the reduced orbit bundle by

$$\tilde{\wp}^{\mathrm{H}}_{[(q,\mathbf{z})]}([(\mathbf{u}, \mathbf{w})]) := [(\wp^{\mathrm{H}}_q(\mathbf{u}), \mathbf{0})]. \tag{11}$$

Here, $(\mathbf{u}, \mathbf{w}) \in T_qQ \oplus T_{\mathbf{z}}\mathrm{orbit}(\mathbf{z}_0)$.[4] Correspondingly, the appropriate vertical projection reads

$$\tilde{\wp}^{\mathrm{V}}_{[(q,\mathbf{z})]}([(\mathbf{u}, \mathbf{w})]) = [(\mathbf{0}, \mathbf{w} + \rho'_{\mathrm{H}}((\iota^*\omega)_q(\mathbf{u}))\mathbf{z})], \tag{12}$$

where $\omega \in \Omega^1(P, \mathrm{Lie}(G))$ is the connection form of A and $\rho'_{\mathrm{H}} \equiv \mathrm{d}\rho_{\mathrm{H}}(e)$ is the "derived representation" of the Lie algebra of G.

Consequently, along $\mathrm{im}(\mathcal{V}) \subset \mathcal{O}\mathrm{rbit}(\mathbf{z}_0)$, we obtain the following identity for a connection on $\mathcal{P}(\mathcal{M}, G)$ compatible with the vacuum $(\mathcal{Q}, \iota)$:

$$\begin{aligned}&\tilde{\wp}^{\mathrm{V}}_{[(q,\mathbf{z}_0)]}([(\mathbf{u}, \mathbf{w})])\\ &\quad = [(\mathbf{0}, \mathbf{w})] = [(\mathbf{u}, \mathbf{w})] - \mathrm{d}\mathcal{V}(\pi_{\mathrm{orb}}([(q, \mathbf{z}_0)]))(\mathrm{d}\pi_{\mathrm{orb}}([(q, \mathbf{z}_0)]))([(\mathbf{u}, \mathbf{w})]).\end{aligned} \tag{13}$$

In other words, when restricted to the vacuum $\mathrm{im}(\mathcal{V})$ any associated reducible connection A looks like the canonical flat connection that is defined by $\mathrm{d}(\mathcal{V} \circ \pi_{\mathrm{orb}})$. In particular, formula (13) implies that for any connection A on $\mathcal{P}(\mathcal{M}, G)$ compatible with the vacuum section $\mathcal{V}$ one obtains

$$\partial_A^{E_{\mathrm{H,red}}}\mathcal{V} = \tilde{\wp}^{\mathrm{V}}_{\mathcal{V}} \circ \mathrm{d}\mathcal{V} \equiv 0. \tag{14}$$

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[3] Note that every connection on the reduced principal H-bundle $\mathcal{Q}(\mathcal{M}, H)$ induces a connection on the principal G-bundle $\mathcal{P}(\mathcal{M}, G)$. But not vice versa, in general. If the latter happens to hold true, the connection is said to be reducible. Clearly, the set of reducible connections on $\mathcal{P}(\mathcal{M}, G)$ is in one-to-one correspondence with the connections on $\mathcal{Q}(\mathcal{M}, H)$.

[4] Notice that $(\mathbf{u}', \mathbf{w}') \sim (\mathbf{u}, \mathbf{w})$ if and only if there exists $h \in H$ and $\eta \in \mathrm{Lie}(H)$, such that $T_{qh}Q \ni \mathbf{u}' = \mathrm{d}\mathcal{R}_h(q)(\mathbf{u} - \mathrm{d}/\mathrm{d}t(q\exp(t\mathrm{ad}_h(\eta)))|_{t=0})$ and $T_{\rho_{\mathrm{H}}(h^{-1})\mathbf{z}}\mathrm{orbit}(\mathbf{z}_0) \ni \mathbf{w}' = \rho(h^{-1})(\mathbf{w} + \rho'_{\mathrm{H}}(\mathrm{ad}_h(\eta))\mathbf{z})$.

In contrast, a vacuum pair $(\Xi, \mathcal{V})$ geometrically represents a minimum of the energy of a Yang-Mills–Higgs gauge theory. It thus generalizes the canonical vacuum pair $(\mathrm{d}, \mathbf{z}_0)$, usually referred to in particle physics. The following shows in what sense the canonical vacuum pair is unique (up to gauge equivalence). In fact, the existence of vacuum pairs relates the topology of spacetime $\mathcal{M}$ to that of $\mathcal{P}(\mathcal{M}, G)$.

Proposition 2.1. *Let again* $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{H}}, V_{\mathrm{H}})$ *be the data defining a Yang-Mills–Higgs gauge theory. Also, let* $\mathbf{z}_0$ *be some minimum of a general Higgs potential* V_{H}. *If spacetime is simply connected, then there exists (up to gauge equivalence) at most one vacuum pair in* $\mathcal{A}(\xi_{\mathrm{H}}) \times \Gamma(\xi_{\mathrm{H}})$ *with respect to the chosen minimum.*

Proof. Let $\pi_1(\mathcal{M}) = 0$. Then, $\mathcal{P}(\mathcal{M}, G)$ possesses a flat connection iff the principal G-bundle is equivalent to $\mathcal{M} \times G \stackrel{\mathrm{pr}_1}{\to} \mathcal{M}$. Moreover, the flat connection is equivalent to the canonical connection on the trivial principal G-bundle (cf. Chapter 9.2, Proposition 9.2 in [6]). Thus, up to equivalence we may assume that $\mathcal{P}(\mathcal{M}, G)$ is trivial. Of course, the same holds true for any vacuum that possesses a flat connection. Since the embedding is right equivariant we obtain

$$\begin{array}{ccc} \mathcal{M} \times \mathrm{H} & \xrightarrow{\ \iota\ } & \mathcal{M} \times \mathrm{G} \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & \mathcal{M} & \end{array}$$

where $\iota(x, h) = (x, \gamma(x)h)$ and $\gamma \in \mathcal{C}^\infty(\mathcal{M}, G)$. Consequently, if there exists a vacuum pair $(\partial, \mathcal{V})$ it must be gauge equivalent to $(\mathrm{d}, \mathbf{z}_0)$. □

Notice that nontrivial vacua may exist even if spacetime is simply connected. The notion of vacuum pairs is clearly more restrictive than that of vacua.

So far we have discussed a minimum of the energy of a Yang-Mills–Higgs gauge theory from the perspective of Yang-Mills–Higgs pairs. Next we will show how the notion of a vacuum pair can be used to "globalize" the *bosonic mass matrices*. For this let $\mathbb{K} = \mathbb{R}$. In the case where $\mathbb{K} = \mathbb{C}$, we regard the Higgs bundle as a real vector bundle of rank $2N$. Accordingly, in what follows the general Higgs potential is considered as a real function and ρ_{H} denotes an orthogonal representation of G (the real form of a unitary representation).

Definition 2.3. Let $(\mathcal{Q}, \iota)$ be a vacuum with respect to a minimum $\mathbf{z}_0 \in \mathbb{R}^N$ of a general Higgs potential V_{H}. The *global mass matrix of the Higgs boson* is the section $\mathcal{V}^* M_{\mathrm{H}}^2 \in \Gamma(\xi_{\mathrm{End}(E_{\mathrm{H}})})$ defined by the equivariant mapping:

$$\nu^* M_{\mathrm{H}}^2 : P \to \mathrm{End}(\mathbb{R}^N), \qquad p = \iota(q)g \mapsto \rho_{\mathrm{H}}(g^{-1}) M_{\mathrm{H}}^2(\mathbf{z}_0) \rho_{\mathrm{H}}(g). \tag{15}$$

Here, $M_{\mathrm{H}}^2(\mathbf{z}_0) \in \mathrm{End}(\mathbb{R}^N)$ is given by $M_{\mathrm{H}}^2(\mathbf{z}_0)\mathbf{z} \cdot \mathbf{z}' := \mathcal{H}\mathrm{ess}(V_{\mathrm{H}})(\mathbf{z}_0)(\mathbf{z}, \mathbf{z}')$ for all $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^N$. The equivariant mapping $\nu \in \mathcal{C}^\infty_{\rho\text{-eq}}(P, \mathrm{orbit}(\mathbf{z}_0))$ corresponds to the vacuum section of $(\mathcal{Q}, \iota)$, i.e. $\nu(P) = \rho_{\mathrm{H}}(g^{-1})\mathbf{z}_0$ for all $p = \iota(q)g \in P$.

Notice that with respect to a vacuum pair $(\Xi, \mathcal{V})$, we may identify the affine set of all (principal) connections on $\mathcal{P}(\mathcal{M}, G)$ with $\xi_{\mathrm{ad}(P)}$. The latter can in turn be identified with the bundle ξ_{YM}:

$$\pi_{\mathrm{YM}} : E_{\mathrm{YM}} := Q \times_H \mathrm{Lie}(G) \to \mathcal{M}. \tag{16}$$

We call the bundle $\tau^*_{\mathrm{M}} \otimes \xi_{\mathrm{YM}}$ the *Yang-Mills bundle* and interpret it as the geometrical analog of a "real" gauge boson.[5]

Definition 2.4. The *global mass matrix of the gauge boson* is the section $\mathcal{V}^* M^2_{\mathrm{YM}} \in \Gamma(\xi_{\mathrm{End(ad}(P))})$ defined by the equivariant mapping:

$$\nu^* M^2_{\mathrm{YM}} : P \to \mathrm{End}(\mathrm{Lie}(G)), \qquad p = \iota(q) \mapsto \mathrm{ad}_{g^{-1}} \circ M^2_{\mathrm{YM}}(\mathbf{z}_0) \circ \mathrm{ad}_g. \tag{17}$$

Here, $M^2_{\mathrm{YM}}(\mathbf{z}_0) \in \mathrm{End}(\mathrm{Lie}(G))$ is defined by $\beta(M^2_{\mathrm{YM}}(\mathbf{z}_0)\eta, \eta') := 2\rho'_{\mathrm{H}}(\eta)\mathbf{z}_0 \cdot \rho'_{\mathrm{H}}(\eta')\mathbf{z}_0$ for all $\eta, \eta' \in \mathrm{Lie}(G)$. The ad-invariant bilinear form β denotes the most general Killing form on $\mathrm{Lie}(G)$ parameterized by the "Yang-Mills coupling constants".

Though defined with respect to a vacuum pair the spectrum of the bosonic mass matrices is constant throughout $\mathcal{M}$ and only depends on the orbit of the minimum $\mathbf{z}_0$. Moreover, both sections $\mathcal{V}^* M^2_{\mathrm{H}}, \mathcal{V}^* M^2_{\mathrm{YM}}$ commute with the invariance group of the vacuum pair. This proves the following lemma.

Lemma 2.1. *Let* $(\Xi, \mathcal{V})$ *be a vacuum pair of a spontaneously broken Yang-Mills–Higgs gauge theory. The Higgs boson and the gauge boson represented, respectively, by* $\xi_{\mathrm{H,red}}$ *and by* ξ_{YM} *decompose into "bosons of mass m"*:

$$\xi_{\mathrm{H,red}} = \bigoplus_{m^2_{\mathrm{H}} \in \mathrm{spec}(M^2_{\mathrm{H}})} \xi_{\mathrm{H},m^2_{\mathrm{H}}}, \tag{18}$$

$$\xi_{\mathrm{YM}} = \bigoplus_{m^2_{\mathrm{YM}} \in \mathrm{spec}(M^2_{\mathrm{YM}})} \xi_{\mathrm{YM},m^2_{\mathrm{YM}}}. \tag{19}$$

Here, $\xi_{\mathrm{H},m^2_{\mathrm{H}}}$ *and* $\xi_{\mathrm{YM},m^2_{\mathrm{YM}}}$ *denote the appropriate eigenbundles of* $\mathcal{V}^* M^2_{\mathrm{H}}$ *and of* $\mathcal{V}^* M^2_{\mathrm{YM}}$, *respectively*.

Notice that this decomposition explicitly refers to a vacuum pair. However, the rank of $\xi_{\mathrm{H},m^2_{\mathrm{H}}}, \xi_{\mathrm{YM},m^2_{\mathrm{YM}}}$ only depends on the orbit of $\mathbf{z}_0$ and is thus independent of the vacuum pair chosen.

In the next section, we will discuss another decomposition of the Higgs bundle geometrically representing the splitting of the Higgs boson into the "Goldstone boson" and the "physical Higgs boson". The rank of the corresponding vector bundles equals the rank of $\mathcal{V}^* M^2_{\mathrm{YM}}$ and of $\mathcal{V}^* M^2_{\mathrm{H}}$. This permits a geometrical interpretation of the so-called "Higgs Dinner". We discuss its dependence on vacuum pairs and how the latter are related to the "unitary gauge".

[5] τ^*_{M} denotes the cotangent bundle. Sometimes we will omit the spin degrees of freedom and refer to the "internal bundle" ξ_{YM} as the gauge boson. In contrast to a real gauge boson, a connection on $\mathcal{P}(\mathcal{M}, G)$ is interpreted as the geometrical analog of a "virtual" gauge boson.

3. The "Higgs Dinner"

In this section, we discuss the Higgs–Kibble mechanism ("Higgs Dinner") from a geometrical perspective. For this, we first translate Goldstone's theorem into geometrical terms and then show how the Higgs Dinner is related to the notion of vacuum pairs. In particular, we want to stress that the existence of the so-called "unitary gauge" is not necessary for the existence of the Higgs Dinner, cf. Chapter 10.3 in [1].

Let $\mathbf{z}_0 \in \mathbb{K}^N$ be a minimum of a general Higgs potential $V_{\rm H}$. In what follows, we will mainly be interested in the real case $\mathbb{K} = \mathbb{R}$. Thus, if $\mathbb{K} = \mathbb{C}$ we will consider the real form of the unitary representation $\rho_{\rm H}$ and take the Higgs bundle $\xi_{\rm H}$ as a real vector bundle of rank $2N$. Likewise, we will regard the Higgs potential as a real function. Let again $H = I(\mathbf{z}_0)$ be the isotropy group of the chosen minimum $\mathbf{z}_0 \in \mathbb{R}^N$ and $\mathrm{Lie}(H)^\perp \subset \mathrm{Lie}(G)$ the orthogonal complement of $\mathrm{Lie}(H)$ with respect to the Killing form $\kappa_{\rm G}$ on G. We then consider the following two subspaces of $\mathbb{R}^N$:

$$W_{\rm G} := \{\mathbf{z} \in \mathbb{R}^N | \mathbf{z} = T\mathbf{z}_0,\, T \in \rho'_{\rm H}(\mathrm{Lie}(H)^\perp) \subset \mathrm{so}(N)\}, \tag{20}$$

$$W_{\rm H,phys} := W_{\rm G}^\perp. \tag{21}$$

Since $H \subset G$ is a closed subgroup, it follows that both the *Goldstone space* $W_{\rm G}$ and the *physical Higgs space* $W_{\rm H,phys}$ are H-invariant subspaces of $\mathbb{R}^N$. As a consequence, one may associate with a vacuum $(\mathcal{Q}, \iota)$ the two real vector bundles $\xi_{\rm G}, \xi_{\rm H,phys}$ defined by

$$\pi_{\rm G} : E_{\rm G} := \mathcal{Q} \times_{\rho_{\rm G}} W_{\rm G} \to \mathcal{M}, \tag{22}$$

$$\pi_{\rm H,phys} : E_{\rm H,phys} := \mathcal{Q} \times_{\rho_{\rm H,phys}} W_{\rm H,phys} \to \mathcal{M}. \tag{23}$$

Here, respectively, $\rho_{\rm G} := \rho_{\rm H}|_{W_{\rm G}}$, $\rho_{\rm H,phys} := \rho_{\rm H}|_{W_{\rm H,phys}}$ denotes the restrictions of $\rho_{\rm H}$ to the Goldstone and the physical Higgs space (20) and (21) with respect to the subgroup H. For instance, $\rho_{\rm G}(h) := \rho_{\rm H}(h)|_{W_{\rm G}}$ for all $h \in H$. We have thus proved the following lemma.

Lemma 3.1. *Let* $(\mathcal{P}(\mathcal{M}, G), \rho_{\rm H}, V_{\rm H})$ *be the data of a Yang-Mills–Higgs gauge theory. Also let* $(\mathcal{Q}, \iota)$ *be a vacuum with respect to some minimum* $\mathbf{z}_0 \in \mathbb{K}^N$ *of* $V_{\rm H}$. *Provided that* $N + \dim(H) - \dim(G) \geq 0$ *the reduced Higgs bundle* $\xi_{\rm H,red}$ (*considered as a real vector bundle*) *is* $\mathbb{Z}_2$*-graded*:

$$\xi_{\rm H,red} = \xi_{\rm G} \oplus \xi_{\rm H,phys}, \tag{24}$$

where, respectively, $\xi_{\rm G}$ *and* $\xi_{\rm H,phys}$ *denote the Goldstone and the physical Higgs bundle with respect to the chosen vacuum.*

Note that

$$\mathrm{rk}(\xi_{\rm H,phys}) = \dim(\mathrm{im}(\mathcal{V}^* M_{\rm H}^2)), \tag{25}$$

$$\mathrm{rk}(\xi_{\rm G}) = \dim(\ker(\mathcal{V}^* M_{\rm H}^2)). \tag{26}$$

Correspondingly, the rank of the Goldstone and the physical Higgs bundle only depends on the orbit of $\mathbf{z}_0$ and not of the chosen vacuum $(\mathcal{Q}, \iota)$.

The geometrical meaning of the Goldstone bundle is as follows: let $\mathcal{V} \in \Gamma(\xi_{\rm H})$ be the vacuum section that corresponds to $(\mathcal{Q}, \iota)$. Then, we have the isomorphism ($x \in \mathcal{M}$):

$$E_{\rm G}|_x \simeq V_{\mathcal{V}(x)}\mathcal{O}{\rm rbit}(\mathbf{z}_0). \tag{27}$$

Here, $V_{\mathcal{V}(x)}\mathcal{O}{\rm rbit}(\mathbf{z}_0)$ denotes the vertical subspace of the tangent space $T_{\mathcal{V}(x)}\mathcal{O}{\rm rbit}(\mathbf{z}_0)$ along the vacuum section $\mathcal{V}$. Thus, the Goldstone bundle can be identified with the vertical bundle of the orbit bundle along the chosen vacuum section.

The equality (26) can be considered as a geometrical variant of *Goldstone's Theorem* (cf. [4]); there is a massless spin-zero boson if the gauge symmetry is spontaneously broken. However, by interacting with the gauge boson the Goldstone boson physically manifests itself as the "longitudinal component" of certain massive spin-one bosons. This is known to be the Higgs–Kibble mechanism (cf. [5]). In fact, we obtain

$$\mathrm{rk}(\xi_{\rm G}) = \dim(\mathrm{im}(\mathcal{V}^* M^2_{\rm YM})) \tag{28}$$

and the massive vector bosons which the Higgs Dinner refers to, are geometrically represented by the eigenbundles (19) of $\mathcal{V}^* M^2_{\rm YM}$. Notice that if $\mathcal{P}(\mathcal{M}, G)$ is supposed to be nontrivial there must be at least one (massless) gauge boson.

Usually, the Higgs Dinner assumes the existence of a specific gauge, called the *unitary gauge*. It is assumed that an equivariant mapping $\gamma \in \mathcal{C}^\infty_{\text{Ad-eq}}(P, G)$ exists for every $\Phi \in \Gamma(\xi_{\rm H})$, such that $\gamma(P)^{-1}\phi(P)$ is orthogonal to the Goldstone space $W_{\rm G}$ for all $p \in P$. Here, $\phi \in \mathcal{C}^\infty_{\rho\text{-eq}}(P, \mathbb{K}^N)$ is the equivariant mapping which corresponds to the section Φ. For this reason, the Goldstone boson is sometimes considered as being "spurious" for it can be "gauged away". Of course, this is misleading because of the manifestation of the Goldstone boson as longitudinal components of massive vector bosons (28). In what follows, we give a geometrical description of both the Higgs Dinner and the unitary gauge and show how they are related to the vacuum chosen.

Definition 3.1. Let $(\mathcal{Q}, \iota)$ be a vacuum with respect to some minimum $\mathbf{z}_0$ and let $\Phi \in \Gamma(\xi_{\rm H})$ be a state of the Higgs boson. We define the Higgs boson to be in the "unitary gauge" with respect to the chosen vacuum iff $\iota^*\Phi \in \Gamma(\xi_{\rm H,phys})$. Here, $\iota^*\Phi(x) := [(q, \iota^*\phi(q))]|_{q\in\pi_Q^{-1}(x)}$ and $\phi \in \mathcal{C}^\infty_{\rho\text{-eq}}(P, \mathbb{K}^N)$ is the corresponding equivariant mapping of Φ.

Of course, one can always obtain such a Φ simply by projecting out the Goldstone part of Φ. However, this raises the question why this can always be done without loss of generality? A sufficient condition is given by the following proposition.

Proposition 3.1. *Let* $(\mathcal{Q}, \iota)$ *be a vacuum with respect to some minimum* $\mathbf{z}_0$ *of a general Higgs potential* $V_{\rm H}$. *Let* $\Phi \in \Gamma(\xi_{\rm H})$ *be a state of the Higgs boson* (*again, considered as a real vector bundle*). *If the mapping*

$$F_\phi : P \to \mathrm{Lie}(G)^*, \qquad p \mapsto \begin{cases} \mathrm{Lie}(G) & \to \quad \mathbb{R} \\ \eta & \mapsto \quad \rho'_{\rm H}(\eta)\mathbf{z}_0 \cdot \phi(P) \end{cases} \tag{29}$$

is of rank $\dim(G) - \dim(H)$ *and* $F_\phi^{-1}(0) \subset \iota(Q) \subset P$, *then* Φ *is in the unitary gauge with respect to the vacuum* $(\mathcal{Q}, \iota)$.

Proof. The local part of the proof is the same as given in [1, Chapter 10.3, Theorem 10.3.10]. The idea goes back to Weinberg [12]. Since G is assumed to be compact the mapping:

$$\Theta_\phi : P \to \mathbb{R}, \qquad p \mapsto \mathbf{z}_0 \cdot \phi(P) \tag{30}$$

has a critical point $p_0 \in \pi_P^{-1}(x)$ for each $x \in \mathcal{M}$, and $F_\phi^{-1}(0) \subset P$ is the critical set of Θ_ϕ. Note that Θ_ϕ is H-invariant and thus descends to a well-defined mapping on the orbit bundle. The rank condition of the proposition then guarantees that $F_\phi^{-1}(0)$ is a smooth sub-manifold of dimension $\dim(\mathcal{M}) + \dim(H)$, which is transversal to each fiber $\pi_P^{-1}(x) \subset P$. Therefore, by the implicit function theorem there exists a family of local trivializations $(U_\alpha, \sigma_\alpha)_{\alpha \in \Lambda}$ of $\mathcal{P}(\mathcal{M}, G)$ (Λ some index set), such that $\mathrm{im}(\sigma_\alpha) \subset F_\phi^{-1}(0)$. As a consequence of the assumption $F_\phi^{-1}(0) \subset \iota(Q)$, the mapping $\mathcal{M} \ni x \mapsto [(\sigma_\alpha(x))] \in \mathcal{O}\mathrm{rbit}(\mathbf{z}_0)$ is well defined and coincides with the vacuum section that corresponds to $(\mathcal{Q}, \iota)$. Let $\iota(q) = \sigma_\alpha(x)$ and $w_{\mathrm{G}} = [(q, T\mathbf{z}_0)] = (x, \mathbf{w}_{\mathrm{G}}) \in E_{\mathrm{G}}$ be arbitrary. We may write $\Phi(x) = [(\sigma_\alpha(x), \phi(\sigma_\alpha(x)))]$ and thus $\langle w_{\mathrm{G}}, \iota^*\Phi(x)\rangle = T\mathbf{z}_0 \cdot \iota^*\phi(q) = 0$. Therefore, $\iota^*\Phi$ is orthogonal to the Goldstone bundle defined with respect to the vacuum $(\mathcal{Q}, \iota)$. □

We call the set $F_\phi^{-1}(0) \subset P$, defined by the mapping (29), the *critical set* associated with a state $\Phi \in \Gamma(\xi_{\mathrm{H}})$ of the Higgs boson. If this critical set defines a sub-manifold of dimension $\dim(\mathcal{M}) + \dim(H)$, then it also defines a vacuum section $\mathcal{V}_\phi \in \Gamma(\xi_{\mathrm{orbit}(\mathbf{z}_0)})$. Clearly, with respect to the corresponding vacuum $(\mathcal{Q}_\phi, \iota_\phi)$ the state Φ is in the unitary gauge. There exists a gauge transformation $f \in \mathrm{Aut}_{\mathrm{eq}}(P)$ such that $f^*\Phi$ is in the unitary gauge with respect to the original vacuum $(\mathcal{Q}, \iota)$ iff the latter is gauge equivalent to $(\mathcal{Q}_\phi, \iota_\phi)$. Note that a necessary condition for the existence of a vacuum, with respect to which a state Φ of the Higgs boson is in the unitary gauge, is that Φ does not vanish. Before discussing a specific class of Higgs potentials, such that $\Phi \in \Gamma(\xi_{\mathrm{H}})\backslash\{\mathcal{O}\}$, with $\mathcal{O}$ being the zero section, is also a sufficient condition for the existence of an appropriate vacuum, we give a simple example clarifying the geometrical idea which underlies the unitary gauge.

For this let $G = U(1)$ and $\mathcal{P}(\mathcal{M}, G)$ be equivalent to the trivial principal $U(1)$-bundle $\mathcal{M} \times U(1) \stackrel{\mathrm{pr}_1}{\to} \mathcal{M}$ (according to the corresponding remark in the last section this holds true, in particular, if all "gauge bosons" are supposed to be massive). Let $N = 1$ and the representation ρ_{H} be the defining one on $\mathbb{C}$. Also let us assume that $V_{\mathrm{H}}(z) := (1 - |z|^2)^2$. In this case, there is only one orbit of minima which can be identified with the one-sphere $S^1 \subset \mathbb{R}^2$. Note that one has to select one minimum $z_0 \in \mathbb{C}$ in order to identify $U(1)$ with S^1 (here, $H = \{1\}$). We also may identify $\Gamma(\xi_{\mathrm{H}})$ with $\mathcal{C}^\infty(\mathcal{M}, \mathbb{C})$ and, correspondingly, $\Gamma(\xi_{\mathrm{orbit}(z_0)})$ with $\mathcal{C}^\infty(\mathcal{M}, S^1)$. Up to equivalence the critical set of a state $\varphi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{C})$ of the Higgs boson reads

$$F_\varphi^{-1}(0) = \{(x, g) \in \mathcal{M} \times U(1) | Tz_0 \cdot g^{-1}\varphi(x) = 0\} \subset P. \tag{31}$$

Here, $T \in \mathrm{so}(2)$ is the real form of the generator of $U(1)$. In the case at hand the fiber derivative of the mapping (29) can be identified with the (pointwise) bilinear form:

$$\mathcal{F}F_\varphi : P \times \mathbb{R}^2 \to \mathbb{R}, \qquad (p = (x, g), (\lambda, \lambda')) \mapsto -\lambda\lambda' z_0 \cdot g^{-1}\varphi(x). \tag{32}$$

Therefore, if $\varphi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{C}\backslash\{0\})$, then the critical set of φ defines a smooth sub-manifold of P of dimension $\dim(\mathcal{M})$ (since H is trivial). In this case, one can define a gauge transformation by the mapping[6]

$$\gamma : \mathcal{M} \to U(1), \qquad x \mapsto g, \tag{33}$$

where $g \in \mathrm{pr}_1^{-1}(x) \cap F_\varphi^{-1}(0)$. Indeed, in the particular case at hand the critical set of a nonvanishing state can be considered as the graph of the unitary gauge transformation (33). The corresponding vacuum section $\mathcal{V}_\varphi$ is given by $\mathcal{V}_\varphi(x) := (x, \gamma(x)z_0)$ which obviously is gauge equivalent to the canonical one. Finally, the vacuum $(\mathcal{Q}_\varphi, \iota_\varphi)$ may be identified with the embedding

$$\mathcal{M} \to \mathcal{M} \times U(1), \qquad x \mapsto (x, \gamma(x)) \tag{34}$$

which can be considered as an element of the gauge group (unitary gauge transformation). This particularly exhibits the relation between the unitary gauge of a state and the vacuum, geometrically considered as a section in the Higgs bundle.

Of course, the example discussed above is very special in several respects and can also be discussed more straightforwardly. The reason for discussing the above example in some detail is to demonstrate certain geometrical features that can be generalized to less trivial examples. This is what we want to discuss next.

Concerning the existence of the unitary gauge, the basic feature of the above example is that the orbit of any minimum is homeomorphic to a sphere of codimension one. Note that any vacuum section is in the unitary gauge with respect to itself. Thus, a vacuum section generates the physical Higgs bundle, provided the latter is of rank one. Moreover, it is straightforward to see that in the unitary gauge with respect to the vacuum $(\mathcal{Q}_\varphi, \iota_\varphi)$ the given section Φ reads $(x \in \mathcal{M})$[7]:

$$\Phi(x) = \|\Phi(x)\|\mathcal{V}_\varphi(x). \tag{35}$$

Note that $\iota_\varphi^*\Phi(x) = (x, |\varphi(x)|\, z_0) \in E_{\mathrm{H,phys}}|_x$. The basic features of the above example can easily be generalized.

Definition 3.2. We call a general Higgs potential V_{H} "rotationally symmetric" if there exists a smooth real valued function $f_{\mathrm{H}} \in \mathcal{C}^\infty(\mathbb{R}_+)$ such that $V_{\mathrm{H}} = f_{\mathrm{H}} \circ r$. Here, $\mathbb{K}^N \xrightarrow{r} \mathbb{R}_+$, $\mathbf{z} \mapsto |\mathbf{z}|$ denotes the "radius function".

Clearly, most of the examples studied in physics are covered by this class of Higgs potentials. This holds true especially for the (minimal) Standard Model. We have the following proposition.

Proposition 3.2. *Let $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{H}}, V_{\mathrm{H}})$ be the data defining a Yang-Mills–Higgs gauge theory, where the Higgs potential is assumed to be rotationally symmetric. For every nonvanishing state, $\Phi \in \Gamma(\xi_{\mathrm{H}})$ of the Higgs boson there exists a vacuum with respect to which the state is in the unitary gauge.*

[6] Actually, this is a general feature if the symmetry breaking were supposed to be complete.
[7] Note that we have put all physical constants, parameterizing the Higgs potential, equal to 1.

Proof. Since $V_{\rm H}$ is assumed to be rotationally symmetric the orbit of a minimum $\mathbf{z}_0$ can be identified with a sphere $S^{N-1}(r_0) \subset \mathbb{R}^N$ of radius $r_0 := r(\mathbf{z}_0)$. Consequently, we have $\mathrm{rk}(\xi_{\rm H,phys}) = 1$. This holds true for any vacuum $(\mathcal{Q}, \iota)$. In particular, with respect to $\Phi \in \Gamma(\xi_{\rm H})\backslash\{\mathcal{O}\}$ we may define a vacuum $(\mathcal{Q}_\phi, \iota_\phi)$ by

$$\mathcal{V}_\phi : \mathcal{M} \to \mathcal{O}\mathrm{rbit}(\mathbf{z}_0), \qquad x \mapsto \frac{|\mathbf{z}_0|}{\|\Phi(x)\|}\Phi(x). \tag{36}$$

Then, it follows from what we discussed before that Φ is in the unitary gauge with respect to the vacuum $(\mathcal{Q}_\phi, \iota_\phi)$. □

Note that even if $\mathcal{P}(\mathcal{M}, G)$ is trivial the above statement does not necessarily imply the existence of a unitary gauge transformation analogous to (33).

Let again $(\mathcal{P}(\mathcal{M}, G), \rho_{\rm H}, V_{\rm H})$ be the data defining a Yang-Mills–Higgs gauge theory and $(\Xi, \mathcal{V})$ a vacuum pair that spontaneously breaks the gauge symmetry. With respect to the original gauge group $\mathcal{G} = \Gamma(\xi_{\mathrm{Ad}(P)})$, we have the gauge boson geometrically represented by the Hermitian vector bundle $\tau^*_{\rm M} \otimes \xi_{\mathrm{ad}(P)}$ and the Higgs boson by $\xi_{\rm H}$. With respect to the invariance group $\mathcal{H} = \Gamma(\xi_{\mathrm{Ad}(Q)})$ of the vacuum $(\mathcal{Q}, \iota)$ we have, respectively, the gauge boson together with the Goldstone and the physical Higgs boson geometrically represented by the Hermitian vector bundle $\tau^*_{\rm M} \otimes \xi_{\rm YM}$, $\xi_{\rm G}$ and $\xi_{\rm H,phys}$. In addition we consider the vector bundle

$$\mathcal{Q} \times_H \mathrm{Lie}(H)^{\perp} \to \mathcal{M}. \tag{37}$$

This decomposes into the Whitney sum of eigenbundles of $\mathcal{V}^* M^2_{\rm YM}$ like $\xi_{\rm H,phys}$ decomposes into the eigenbundles of $\mathcal{V}^* M^2_{\rm H}$ of nonvanishing masses. Since $W_{\rm G} \simeq \mathrm{Lie}(H)^{\perp}$ the physical Higgs Dinner is geometrically described by the identity

$$\xi_{\mathrm{ad}(Q)} \oplus (\xi_{\rm G} \oplus \xi_{\rm H,phys}) = (\xi_{\mathrm{ad}(Q)} \oplus \xi_{\rm G}) \oplus \xi_{\rm H,phys}. \tag{38}$$

Notice that $\xi_{\mathrm{ad}(Q)} \oplus \xi_{\rm G}$, as a vector bundle, is naturally isomorphic to the Yang-Mills bundle (16) and thus equivalent to $\xi_{\mathrm{ad}(P)}$. Consequently, the Higgs Dinner does not refer to a gauge condition. However, it always refers to a vacuum.

In the last section, we have defined the bosonic mass matrices and called their eigenvalues the "masses" of the bosons which are geometrically represented by the corresponding eigenbundles of the mass matrices. This physical interpretation of the eigenvalues usually refers to the field equation of "free bosons". To also justify this physical interpretation of the eigenvalues within our geometrical description we give the following definition.

Definition 3.3. Let $0 \leq t \leq 1$. A family of Yang-Mills–Higgs pairs $(A_t, \Phi_t) \in \mathcal{A}(\xi_{\rm H}) \times \Gamma(\xi_{\rm H})$ is called a "fluctuation" of a vacuum pair $(\Xi, \mathcal{V})$ provided there is $\Phi_{\rm H,phys} \in \Gamma(\xi_{\rm H,phys})$ and $A = A_{\rm H} \oplus A_{\rm G} \in \Omega^1(\mathcal{M}, \mathrm{Lie}(H) \oplus \mathrm{Lie}(H)^{\perp})$ such that

$$\partial_{A_t} = \partial + tA_{\rm H} + t\rho'_{\rm G}(A_{\rm G}) \equiv \partial^{\mathrm{ad}(Q)}_{A_{\rm H},t} + t\rho'_{\rm G}(A_{\rm G}), \tag{39}$$

$$\Phi_t = \mathcal{V} + t\Phi_{\rm H,phys}. \tag{40}$$

Next, we note that the mass matrices $\mathcal{V}^*M^2_{\mathrm{H}}$, $\mathcal{V}^*M^2_{\mathrm{YM}}$ split according to the decomposition of $\xi_{\mathrm{H,red}}$, ξ_{YM}. That is, we have

$$\mathcal{V}^*M^2_{\mathrm{H}} = M^2_{\mathrm{G}} \oplus M^2_{\mathrm{H,phys}}, \tag{41}$$

$$\mathcal{V}^*M^2_{\mathrm{YM}} = M^2_{\mathrm{YM,H}} \oplus M^2_{\mathrm{YM,G}}, \tag{42}$$

where $\dim(\mathrm{im}(M^2_{\mathrm{H,phys}})) = \dim(\mathrm{im}(\mathcal{V}^*M^2_{\mathrm{H}}))$ and $\dim(\mathrm{im}(M^2_{\mathrm{YM,G}})) = \dim(\mathrm{im}(\mathcal{V}^*M^2_{\mathrm{YM}})) = \dim(\mathrm{im}(M^2_{\mathrm{G}}))$.

Proposition 3.3. *Let $(\Xi, \mathcal{V}) \in \mathcal{A}(\xi_{\mathrm{H}}) \times \Gamma(\xi_{\mathrm{H}})$ be a vacuum pair that spontaneously breaks the gauge symmetry of a Yang-Mills–Higgs gauge theory. Also, let (A_t, Φ_t) be a fluctuation of the vacuum. Then, up to order $\mathcal{O}(t^2)$ the Euler–Lagrange equations in terms of the fluctuation read*

$$\delta^{\mathrm{ad}(Q)}\partial^{\mathrm{ad}(Q)}A_{\mathrm{H}} = 0, \tag{43}$$

$$\delta^{E_{\mathrm{G}}}\partial^{E_{\mathrm{G}}}A_{\mathrm{G}} + M^2_{\mathrm{YM,G}}A_{\mathrm{G}} = 0, \tag{44}$$

$$\delta^{E_{\mathrm{H,phys}}}\partial^{E_{\mathrm{H,phys}}}\Phi_{\mathrm{H,phys}} + M^2_{\mathrm{H,phys}}\Phi_{\mathrm{H,phys}} = 0. \tag{45}$$

Here, $\partial^{\mathrm{ad}(Q)}$, $\partial^{E_{\mathrm{G}}}$, $\partial^{E_{\mathrm{H}}}$ denote the induced flat covariant derivatives on $\xi_{\mathrm{ad}(Q)}$, ξ_{G}, $\xi_{\mathrm{H,phys}}$, respectively, and $\delta^{\mathrm{ad}(Q)}$, $\delta^{E_{\mathrm{G}}}$, $\delta^{E_{\mathrm{H}}}$ are the appropriate co-derivatives.

Proof. The proof results from the usual variational calculation where one takes advantage of the orthogonality of the Goldstone and the Higgs bundle and of the fact that the vacuum section is covariantly constant. □

Notice that the fluctuation A is not compatible with the vacuum. Indeed, it is the deviation of (39) from being compatible with the vacuum that gives rise to the nontriviality of $\mathcal{V}^*M^2_{\mathrm{YM}}$. Since the mass matrices commute with the connection, one may use an orthonormal eigenbasis of the bosonic mass matrices whereby the field equations (43)–(45) read

$$\delta\partial A_{\mathrm{H},(k)} = 0, \tag{46}$$

$$\delta\partial A_{\mathrm{G},(l)} + m^2_{\mathrm{YM,G},l}\mathcal{A}_{\mathrm{G},(l)} = 0, \tag{47}$$

$$\delta\partial\Phi_{\mathrm{H,phys},(j)} + m^2_{\mathrm{H,phys},j}\Phi_{\mathrm{H,phys},(j)} = 0, \tag{48}$$

where $k = 1, \dots, \dim(H)$, $l = 1, \dots, \dim(W_{\mathrm{G}})$ and $j = 1, \dots, \dim(W_{\mathrm{H,phys}})$.

The fact that the connection Ξ is flat does not mean that the principal symbols of the respective second order differential operators in (46)–(48) coincide with their symbols. The symbol, however, is the geometrical object that corresponds to the physical quantity of momenta (squared) of the appropriate particle. If $\mathcal{M}$ is simply connected the principal symbol coincides with the symbol and in this case we recover the usual field equations of “free bosons”. In the slightly more general case we call solutions of the field equations (46)–(48) *quasi-free states*. The corresponding line bundles generated by the eigenbasis of the bosonic mass matrices are interpreted as *asymptotic* (*quasi*)-*free bosons*. Of course, the scale on which this interpretation holds is given by the parameter t. Notice that the difference between asymptotic quasi-free and asymptotic free bosons only results from the topology of spacetime. In contrast, the difference between asymptotic free and free bosons results from their "H-charge". For instance, consider the electroweak sector of the Standard Model (see the next section). In this case, the reduced gauge group equals the electromagnetic gauge group. However, the physical Higgs boson turns out to be electrically uncharged and is thus not only insensitive to an Ahoronov-Bohm like effect but can be geometrically represented by a *trivial* Hermitian line bundle. This holds true even in the case where the underlying electromagnetic vacuum $(\mathcal{Q}, \iota)$ is nontrivial.

In the next section, we give a geometrical interpretation of the bosonic mass matrices as "normal sections" of specific sub-manifolds.

4. Bosonic mass matrices and "normal bundles"

Let $(\mathcal{Q}, \iota)$ be again a vacuum and $\mathcal{V} \in \Gamma(\xi_{\text{orbit}(\mathbf{z}_0)})$ be the corresponding vacuum section. We have already mentioned that the Goldstone bundle $\xi_{\text{G}} \subset \xi_{\text{H,red}}$ might be identified with the vertical bundle of $\mathcal{O}\text{rbit}(\mathbf{z}_0)$ along the vacuum section $\mathcal{V}$. Likewise, one may consider the physical Higgs bundle $\xi_{\text{H,phys}} \subset \xi_{\text{H,red}}$ as the "normal bundle" of $\mathcal{O}\text{rbit}(\mathbf{z}_0) \subset E_{\text{H,red}}$ along the vacuum section $\mathcal{V}$. For this, we consider the (reduced) Higgs bundle as a vector bundle over the (reduced) orbit bundle, i.e.

$$\text{pr}_1 : \pi^*_{\text{orb}} E_{\text{H}} \to \mathcal{O}\text{rbit}(\mathbf{z}_0). \tag{49}$$

Along a vacuum section $\mathcal{V}$ one has

$$\pi^*_{\text{orb}} E_{\text{G}} \oplus \pi^*_{\text{orb}} E_{\text{H,phys}} \to \text{im}(\mathcal{V}) \subset \mathcal{O}\text{rbit}(\mathbf{z}_0), \tag{50}$$

where $\pi^*_{\text{orb}} E_{\text{G}} = V\mathcal{O}\text{rbit}(\mathbf{z}_0)|_{\text{im}(\mathcal{V})}$ and the tangent bundle of $\mathcal{O}\text{rbit}(\mathbf{z}_0)$ splits into

$$T\mathcal{O}\text{rbit}(\mathbf{z}_0)|_{\text{im}(\mathcal{V})} = \text{im}(\text{d}\mathcal{V}) \oplus \pi^*_{\text{orb}} E_{\text{G}}. \tag{51}$$

Thus, $\pi^*_{\text{orb}} E_{\text{H,phys}}$ can be considered as the "normal bundle" of the reduced orbit bundle. This permits to recover the well-known geometrical picture of the Goldstone boson as being parallel and the physical Higgs boson as being orthogonal to the orbit (bundle). The geometrical picture also illuminates why the spectrum of the global mass matrix of the Higgs boson is constant, for it can be regarded as the parallel transport of $M^2_{\text{H}}(\mathbf{z}_0)$ along the specified vacuum. The Hessian of a general Higgs potential is constant along the orbit and positive definite transversally. Thus, it does not come as a surprise that the (global) mass matrix of the Higgs boson is related to the extrinsic curvature of the orbit (bundle). This is most easily exhibited in the case of a rotationally symmetric Higgs potential.

For this, let $V_{\text{H}}(\mathbf{z}) = f_{\text{H}}(r(\mathbf{z})) \equiv f_{\text{H}}(r)$ be rotationally symmetric ($\mathbf{z} \in \mathbb{R}^N$). Let $(\mathcal{Q}, \iota)$ be again a vacuum that spontaneously breaks the gauge symmetry defined by $\mathcal{P}(\mathcal{M}, G)$. Also let $\mathcal{V}$ be the appropriate vacuum section on the reduced Higgs bundle. In the case of a rotationally symmetric Higgs potential, the nontrivial part of the (global) mass matrix of the Higgs boson reads

$$M^2_{\mathrm{H,phys}}(x) = f''_{\mathrm{H}}(r_0)e(x)^* \otimes e(x), \tag{52}$$

where $\|\mathcal{V}(x)\|e(x) := \mathcal{V}(x) \in E_{\mathrm{H,phys},x}$, $e(x)^* \in E^*_{\mathrm{H,phys},x}$ the dual vector, and $r_0 \equiv r(\mathbf{z}_0) = \|\mathcal{V}(x)\|$. The spectrum is given by $\mathrm{spek}(M^2_{\mathrm{H,phys}}) = \{f''_{\mathrm{H}}(r_0)\}$ and the mass matrix is related to an appropriate generalization of the *second fundamental form* of $\mathrm{orbit}(\mathbf{z}_0) \subset \mathbb{R}^N$ due to the formula

$$E_{\mathrm{G},x} \times E_{\mathrm{G},x} \to \mathbb{R}, \qquad (\mathbf{u},\mathbf{w}) \mapsto g_{\mathrm{G},x}(M^2_{\mathrm{H}}(\partial_{\mathbf{u}}e)(x),\mathbf{w}) = f''_{\mathrm{H}}(r_0)g_{\mathrm{G},x}(\mathbf{u},\mathbf{w}). \tag{53}$$

Here, g_{G} denotes the Hermitian product on E_{G}, and ∂ is understood as the covariant derivative on the pull-back bundle $\pi^*_{\mathrm{orb}}\xi_{\mathrm{H,red}}$ with respect to the flat connection $\pi^*_{\mathrm{orb}}\Xi$. The formula (53) generalizes the situation where $\mathcal{P}(\mathcal{M}, G)$ is supposed to be the trivial principal G-bundle $\mathcal{M} \times G \overset{\mathrm{pr}_1}{\to} \mathcal{M}$. In this case, the above formula reduces to

$$W_{\mathrm{G}} \times W_{\mathrm{G}} \to \mathbb{R}, \qquad (\mathbf{u},\mathbf{w}) \mapsto M^2_{\mathrm{H}}(\mathbf{z}_0)de(\mathbf{z}_0)\mathbf{u}\cdot\mathbf{w} = f''_{\mathrm{H}}(r_0)\mathbf{u}\cdot\mathbf{w} \tag{54}$$

which can be regarded as the fiber Hessian of the mapping

$$F_{\mathrm{H}} : \mathbb{R}^N\backslash\{0\} \to \mathbb{R}, \qquad \mathbf{z} \mapsto \mathrm{grad}\, V_{\mathrm{H}}(\mathbf{z})\cdot e(\mathbf{z}) = f'_{\mathrm{H}}(r)r. \tag{55}$$

Here, $e(\mathbf{z}) := \mathbf{z}/\|\mathbf{z}\| \in S^{N-1}$. Notice that $F^{-1}_{\mathrm{H}}(0)$ equals the critical set of the Higgs potential and that

$$\mathrm{grad}\, F_{\mathrm{H}}(\mathbf{z}) = M^2_{\mathrm{H}}(\mathbf{z})e(\mathbf{z}). \tag{56}$$

We shall recover a similar formula for the mass matrix of the gauge boson.

Before we proceed with discussing the geometrical meaning of the mass matrix of the gauge boson, we would like to briefly comment on the situation of a general Higgs potential. The main point of a rotationally Higgs potential is that the codimension of the orbit is equal to 1. Thus, the extrinsic geometry of the orbit bundle is determined by the variation of the vacuum section $\mathcal{V}$ considered as a normalized eigensection of the mass matrix of the Higgs boson. This variation in turn is determined by the (pseudo)Riemannian connection that is induced by the flat connection Ξ. More precisely: the connection Ξ admits to lift the (pseudo)Riemannian metric g_{M} to the orbit bundle. This lifted (pseudo)metric determines a corresponding connection, which together with the vacuum section defines the *Weingarten map* of the orbit bundle (along the vacuum section). The Weingarten map, however, determines the extrinsic geometry of any sub-manifold (see, for instance [8]). In the case of a general Higgs potential, the situation is more complicated since the codimension of the orbit is greater than 1 in general. Therefore, in this case one has more than one normal direction and the appropriate normal bundle is nontrivial. Moreover, in general there are no distinguished normal sections determining the extrinsic geometry of the orbit bundle (c.f. loc sit). However, along the vacuum section the eigensections of the mass matrix of the Higgs boson give rise to a particular set of normal sections like in the case of a rotationally symmetric Higgs potential. Therefore, along the vacuum section the extrinsic geometry of the orbit bundle becomes determined by the mass matrix of the Higgs boson.[8]

[8] The intrinsic geometry, of course, is again determined by the (pseudo)metric g_{M}.

To study the geometrical meaning of the mass matrix of the gauge boson, let $(\mathcal{Q}, \iota)$ be again a vacuum which spontaneously breaks the gauge symmetry that is defined by $\mathcal{P}(\mathcal{M}, G)$. Also, let $(\Xi, \mathcal{V})$ be an appropriate vacuum pair and $\nu \in \mathcal{C}^\infty_{\rho\text{-eq}}(P, \mathbb{R}^N)$ be the equivariant mapping that corresponds to $\mathcal{V}$. That is, $\mathcal{V}(x) = [(p, \nu(P))]|_{p\in\pi_P^{-1}(x)} = [(\iota(q), \mathbf{z}_0)]|_{q\in\pi_Q^{-1}(x)}$. Of course, the vacuum section $\mathcal{V} \in \Gamma(\xi_{\mathrm{H}})$ is always in the unitary gauge with respect to itself. In other words, the vacuum section might be considered as a section in $\xi_{\mathrm{H,phys}}$ (where the latter is defined with respect to the vacuum $(\mathcal{Q}, \iota)$). Moreover, the critical set associated with the vacuum section $F_\nu^{-1}(0) \subset P$ coincides with $\iota(Q)$. Since the vacuum section is constant, the tangential mapping of F_ν equals its fiber derivative $\mathcal{F}F_\nu$. The latter in turn coincides with the fiber Hessian of Θ_ν, which reads

$$\mathcal{F}^2\Theta_\nu : VP \times_P VP \to \mathbb{R}, \qquad (p, \eta_1, \eta_2) \mapsto \rho'(\eta_1\eta_2)\mathbf{z}_0 \cdot \nu(P). \tag{57}$$

Therefore, when restricted to the critical set $F_\nu^{-1}(0)$ we obtain

$$\mathrm{d}F_\nu(\iota(q))(\mathbf{w})\eta' = -\tfrac{1}{2}\beta(M^2_{\mathrm{YM}}(\mathbf{z}_0)\eta, \eta') \tag{58}$$

for all $\eta' \in \mathrm{Lie}(G)$. Here, $\eta \in \mathrm{Lie}(G)$ is determined as the vertical part of $\mathbf{w} \in T_{\iota(q)}P$ with respect to the connection Ξ. Notice that (58) is nonzero iff $\eta, \eta' \in \mathrm{Lie}(H)^\perp \simeq W_{\mathrm{G}}$.

Like in the case of the Higgs bundle, we may consider the adjoint bundle as a vector bundle over P. With respect to a given vacuum section this bundle decomposes as

$$\pi_P^*\mathrm{ad}(Q) \oplus \pi_P^* E_{\mathrm{G}} \to F_\nu^{-1}(0) \subset P. \tag{59}$$

Notice that a general element of $\pi_P^*\mathrm{ad}(Q) \oplus \pi_P^* E_{\mathrm{G}}$ reads $(p = \iota(q), \tau, \rho'(\eta)\mathbf{z}_0)$, where $\tau \in \mathrm{Lie}(H)$ and $\eta \in \mathrm{Lie}(H)^\perp$.

When restricted to $F_\nu^{-1}(0)$ the tangent bundle of P splits into

$$TP|_{F_\nu^{-1}(0)} = TF_\nu^{-1}(0) \oplus \pi_P^* E_{\mathrm{G}}. \tag{60}$$

Thus, $\pi_P^* E_{\mathrm{G}} \to F_\nu^{-1}(0)$ can be regarded as the "normal" bundle of $F_\nu^{-1}(0) = \iota(Q) \subset P$. Consequently, any tangent vector $\mathbf{w} \in T_{\iota(q)}P$ decomposes as $\mathbf{w} = \mathrm{d}\iota(q)\mathbf{u} + \mathbf{w}_{\mathrm{G}}$, where $\mathbf{w}_{\mathrm{G}} \in \pi_P^* E_{\mathrm{G}}|_{\iota(q)}$ and $\mathbf{u} \in T_qQ$.

There is a natural fiber metric (also denoted by β) on the bundle (59), such that $(\pi_P^*\mathrm{ad}(Q) \oplus \pi_P^* E_{\mathrm{G}})|_{\iota(q)}$ is isometric to $(\mathrm{Lie}(G), \beta)$. For each direction $w = (\iota(q), \mathbf{w}) \in TP|_{F_\nu^{-1}(0)}$, we define the "gradient" of F_ν by the relation

$$\beta(\mathrm{grad}\, F_\nu(\iota(q))(\mathbf{w}), \varsigma) := \mathrm{d}F_\nu(\iota(q))(\mathbf{w})\varsigma \tag{61}$$

for all $\varsigma \in (\pi_P^*\mathrm{ad}(Q) \oplus \pi_P^* E_{\mathrm{G}})|_{\iota(q)}$. Then, the nontrivial part of the mass matrix of the gauge boson reads

$$\mathrm{grad}\, F_\nu(w_{\mathrm{G}}) = -\tfrac{1}{2}\nu^* M^2_{\mathrm{YM,G}} w_{\mathrm{G}} \tag{62}$$

which is analogous to (56).

Let $(\eta_1, \ldots, \eta_{\dim(W_{\mathrm{G}})}) \in \mathrm{Lie}(H)^\perp$ be a κ_{G} orthonormal eigenbasis of the nontrivial part of $M^2_{\mathrm{YM}}(\mathbf{z}_0)$. Correspondingly, let $w_{\mathrm{G},1}, \ldots, w_{\mathrm{G},\dim(W_{\mathrm{G}})} \in TP|_{F_\nu^{-1}(0)}$. Then,

$$\mathrm{grad}\, F_\nu(w_{\mathrm{G},l}) = -\tfrac{1}{2}m^2_{\mathrm{YM,G},l} w_{\mathrm{G},l} \tag{63}$$

and we obtain the known formula

$$m^2_{\mathrm{YM,G},l} = 2g^2_{\mathrm{phys},l}g_{\mathrm{G}}(w_{\mathrm{G},l}, w_{\mathrm{G},l}). \tag{64}$$

If G is not simple, the "physical coupling constant" $g_{\mathrm{phys},l}$, in general, is a fractional function of the Yang-Mills coupling constants depending on $\mathrm{Lie}(H) \subset \mathrm{Lie}(G)$. In the case of our previous example, where $G = U(1)$ and $H = \{1\}$, we obtain the usual formula for the "massive photon" $m = \sqrt{2}g_{\mathrm{phys},}|z_0|$, where g_{phys}, is identified with the electric charge.

We have shown in this section that the bosonic mass matrices geometrically correspond to "normal sections" ("gradients") along the vacuum. Here, the vacuum is considered as a sub-manifold either of $\mathcal{O}\mathrm{rbit}(\mathbf{z}_0)$ or of P. In the following section, we come back to the unitary gauge. We discuss its existence in the case of the structure group of the electroweak sector of the (minimal) Standard Model. We are aware that like in the example of $G = U(1)$, this can be achieved in a more straightforward way than presented in the next section. However, we again want to put emphasis on the geometrical background.

5. $G = \mathrm{SU}(2) \times U(1)$

In the preceding section, we discussed the existence of the unitary gauge in the case of the electromagnetic gauge group. In this section, we present an analogous analysis for the more realistic case of the electroweak gauge group of the bosonic part of the Standard Model.

Let $(\mathcal{M}, g_{\mathrm{M}})$ be an arbitrary spacetime. The bosonic part of the Standard Model is fixed by the Yang-Mills–Higgs gauge theory $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{H}}, V_{\mathrm{H}})$, where $G := \mathrm{SU}(2) \times U(1)$ is the well-known structure group of the electroweak sector of the Standard Model. To simplify the notation we again put all physical parameters equal to 1. Up to an additive constant the Higgs potential has the usual form $V_{\mathrm{H}}(\mathbf{z}) := (1 - |\mathbf{z}|^2)^2$, where $\mathbf{z} \in \mathbb{C}^2$. The representation ρ_{H} is defined by $\rho_{\mathrm{H}}(g_{(2)}, g_{(1)})\mathbf{z} := g_{(2)}g_{(1)}\mathbf{z} = g_{(1)}g_{(2)}\mathbf{z}$, where $g_{(1)} \in U(1)$ and $g_{(2)} \in \mathrm{SU}(2)$.

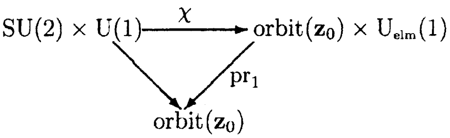
The set of minima of V_{H} is equal to the three-sphere $S^3 \subset \mathbb{R}^4$. On the one hand, when distinguishing a point $\mathbf{z}_0 \in S^3$, we may identify $(S^3, \mathbf{z}_0)$ with the group $\mathrm{SU}(2)$. On the other hand, we may also identify $(S^3, \mathbf{z}_0)$ with $\mathrm{orbit}(\mathbf{z}_0)$. In fact, the isotropy group of an arbitrary minimum $\mathbf{z}_0$, which is isomorphic to $H \equiv U_{\mathrm{elm}}(1)$, is generated by $\tau + i \in \mathrm{Lie}(G) = \mathrm{su}(2) \oplus u(1)$. Note that $\tau \in \mathrm{su}(2) \simeq \mathbb{R}^3 \subset \mathbb{H}$ $(\tau^2 = -1)$ depends on the chosen minimum $\mathbf{z}_0$. Geometrically, each minimum of the Higgs potential permits to distinguish a circle $S^1 \subset S^3 \subset \mathbb{H}$, and the right action of $H \subset G$ on the electroweak structure group is given by

$$\begin{aligned}&(\mathrm{SU}(2) \times U(1)) \times U_{\mathrm{elm}}(1) \to \mathrm{SU}(2) \times U(1),\\ &((g_{(2)}, g_{(1)}), h) \mapsto (g_{(2)}h_{(2)}, g_{(1)}h_{(1)}).\end{aligned} \tag{65}$$

Here, we made use of the fact that every element $h \in U_{\mathrm{elm}}(1)$ decomposes as $h = h_{(2)}h_{(1)} = h_{(1)}h_{(2)}$, where $h_{(2)} := \exp(\tau\theta) \in \mathrm{SU}(2)$ and $h_{(1)} := \exp(\mathrm{i}\theta) \in U(1)$ $(\theta \in [0, 2\pi[)$. As a consequence, $(g_{(2)}, g_{(1)})$ is equivalent to $(g_{(2)}h^{-1}_{(2)}, 1)$, where $h_{(2)} := \exp(\tau\theta)$ for $g_{(1)} = \exp(\mathrm{i}\theta)$. Therefore, we may identify $G/H \simeq \mathrm{orbit}(\mathbf{z}_0)$ with $\mathrm{SU}(2) \simeq (S^3, \mathbf{z}_0)$. Moreover, we have the following principal $U_{\mathrm{elm}}(1)$-bundle

$$G = \mathrm{SU}(2) \times U(1) \to \mathrm{orbit}(\mathbf{z}_0), \qquad (g_{(2)}, g_{(1)}) \mapsto g_{(2)}h^{-1}_{(2)}\mathbf{z}_0. \tag{66}$$

The crucial point is that this bundle is actually trivial. We have the following bundle isomorphism:



which is given by $\chi(g_{(2)}, g_{(1)}) := (g_{(2)}h_{(2)}^{-1}\mathbf{z}_0, h := h_{(2)}h_{(1)})$, where $h_{(1)} := g_{(1)}$.

From the preceding section, we know that a nonvanishing state $\Phi \in \Gamma(\xi_{\mathrm{H}})$ of the Higgs boson is always in the unitary gauge with respect to the vacuum $(\mathcal{Q}_\phi, \iota_\phi)$. Let us then suppose that $\mathcal{P}(\mathcal{M}, G)$ is equivalent to the trivial principal G-bundle. Because of the triviality of the principal $U_{\mathrm{elm}}(1)$-bundle (66), one can lift the corresponding vacuum section $\mathcal{V}_\phi$ to the mapping

$$\gamma : \mathcal{M} \to \mathrm{SU}(2) \times U(1), \qquad x \mapsto \chi^{-1}(\nu_\phi(x), 1) \tag{67}$$

such that $\mathcal{V}_\phi$ is gauge equivalent to the canonical vacuum section. Here, $\mathcal{V}_\phi(x) = (x, \nu_\phi(x))$ with $\nu_\phi \in \mathcal{C}^\infty(\mathcal{M}, \mathrm{orbit}(\mathbf{z}_0))$ and $\iota_\phi^*\Phi(x) = (x, \|\Phi(x)\|\mathbf{z}_0) \in E_{\mathrm{H,phys}}|_x$. Of course, $E_{\mathrm{H,phys}}$ is defined with respect to $(\mathcal{Q}_\phi, \iota_\phi)$ and the embedding ι_ϕ is defined by (67).

The mapping (67) defines the unitary gauge transformation similar to the case of $G = U(1)$ discussed in the last section. Indeed, the triviality of $U(1) \to \mathrm{orbit}(\mathbf{z}_0)$ follows immediately from $H = \{1\}$ and the identification of $U(1)$ with $(S^1, \mathbf{z}_0) \simeq \mathrm{orbit}(\mathbf{z}_0)$. Notice that in both examples the lifting property is independent of the topology of $\mathcal{M}$. In general, if both $\mathcal{P}(\mathcal{M}, G)$ and $G(\mathrm{orbit}(\mathbf{z}_0), I(\mathbf{z}_0))$ are trivial, then up to gauge equivalence there exists only one vacuum $(\mathcal{Q}, \iota)$ with respect to a given minimum $\mathbf{z}_0$. In particular, this vacuum is trivial (i.e. $\mathcal{Q}(\mathcal{M}, H)$ is also trivial). On the other hand, if we assume spacetime to be simply connected we know that the existence of vacuum *pairs* is equivalent to the triviality of $\mathcal{P}(\mathcal{M}, G)$. When we fix a minimum $\mathbf{z}_0$, all vacuum pairs $(\partial, \mathcal{V})$ are gauge equivalent to $(\mathrm{d}, \mathbf{z}_0)$. In this case, only those vacuum sections $\mathcal{V}$ are permitted that give rise to a lift similar to (67). In the particular case of $\mathcal{V}_\phi$ this hold true, iff $\mathcal{Q}_\phi(\mathcal{M}, H)$ is also trivial.

To summarize, if $\mathcal{P}(\mathcal{M}, G)$ is trivial, then a necessary condition for gauge inequivalent vacua to exist with respect to a given minimum $\mathbf{z}_0$ is that the principal $I(\mathbf{z}_0)$-bundle $G(\mathrm{orbit}(\mathbf{z}_0), I(\mathbf{z}_0))$ is nontrivial. Whether this condition is also sufficient depends on the topology of spacetime.

6. Summary and outlook

We geometrically described the possible ground states of the Higgs boson as sections in the orbit bundle, which is associated with the data of a general Yang-Mills–Higgs gauge theory. The notion of vacuum pairs has been used to geometrically describe the Higgs–Kibble mechanism and the unitary gauge. We also gave a necessary and sufficient condition for the existence of the unitary gauge in the case of rotationally symmetric Higgs potentials. The notion of vacuum pairs also permits a geometrical interpretation of the bosonic mass matrices and the physical notion of “free” bosons also within the frame of gauge theories. Moreover, since the notion of vacuum pairs geometrically generalize $(\mathrm{d}, \mathbf{z}_0)$ in the case of the trivial principal G-bundle, it permits to relate the notion of mass to the topology of spacetime. We gave a necessary and sufficient condition for the existence of vacuum pairs in the case where $\pi_1(\mathcal{M}) = 0$. This case turned out to be particularly restrictive. It would be interesting to also study less restrictive spacetime topologies giving rise to gauge inequivalent vacuum pairs.

From a geometrical perspective, we have seen how the masses of the bosons are related to “normal vector fields” of sub-manifolds which are determined by the vacuum. Likewise, it can be shown that the masses of the fermions together with the curvature of spacetime, determine the “intrinsic curvature” of the bundles which geometrically represent “free fermions”. This will be discussed within the geometrical frame of generalized Dirac operators in a forthcoming paper.

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